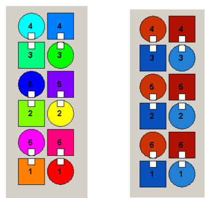
The Contra Cube

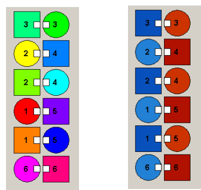
Chris Page's article *The twenty-four duple minor formations* gives a very clear classification of the 24 possible arrangements of dancers on hands four spots and the transitions between them. These arrangements fall into three natural families *Improperish, Properish* and *Diagonal* which are closed under the commonly used symmetric choreography.

Formations

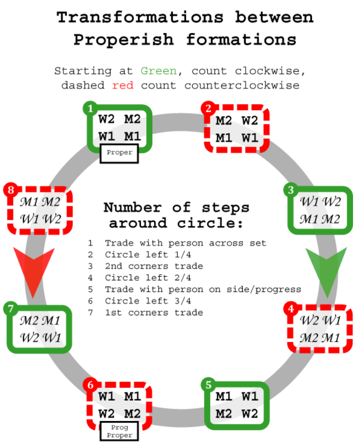
The formations described in Chris' article are permutations of the symbols M1, W1, M2, W2. An equivalent view point in the initial diagramming of a dance uses the same color for all dancers of the same type. To the right is an improper set up with six couples and a coloring, so the arrangement of colors in each of the hands four stations look identical.

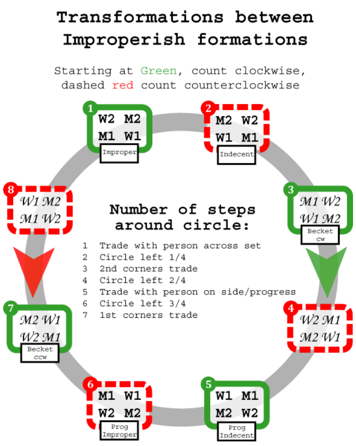


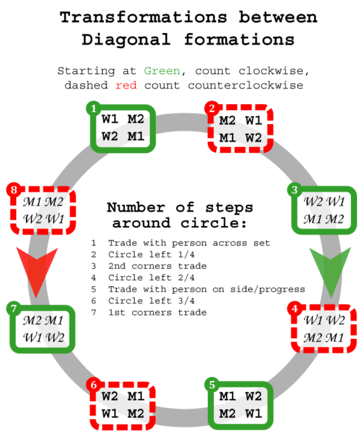
Part way thru the dance (which has some out of set action) the dancers find themselves as pictured, but all hands four stations have colors arranged the same and the coloring has 'circled left ¼'



Examining a specific hands four station, how the four labels or colors move give the permutation corresponding to a given dance move[1]. Chris produced following three figures classifying what might be observed and showing how contra dance moves 1 to 7 give transformations between the eight formations.



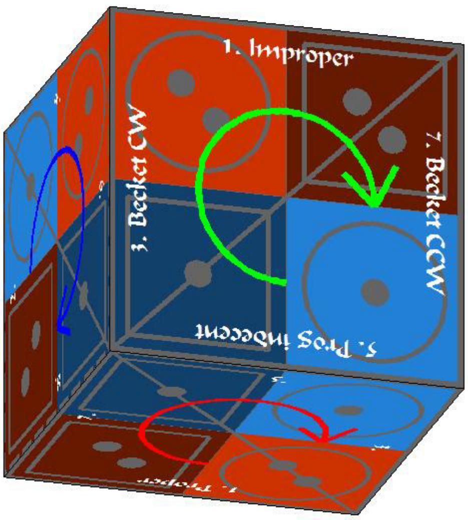


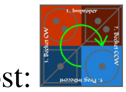


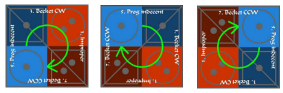
[1] A similar idea was used to understand grid squares where a coloring of the eight dancers in the squares gives an induced permutation.

Rotating a "Contra Cube" give Chris's formations

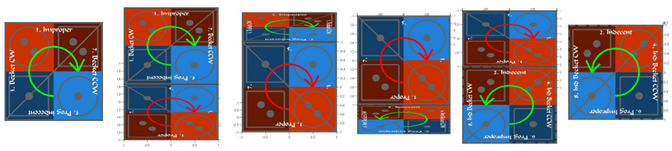
Rotations of a cube allow the formation families and transitions between them to be visualized. The opposing faces relate to the solid and dashed formations in Chris' diagrams.



To use this cube set the initial formation uppermost:  The seven contra dance moves are all rotations of the cube. The circling moves (2,4 and 6) are just the same circling of the cube around a vertical axis finishing at:

 respectively.

A trading move such as 1 (trading across the set), is found by using a rotation axis across the cube giving



Note that half way through a trading move the top face of another formation family appears. See the video at: http://youtu.be/zVA0O791deg

Solid and Dashed Lines vs. Clockwise and Anticlockwise Oriented Faces

In Chris's diagrams the formations outlined with solid and dashed lines correspond opposite faces of the cube. The faces of the cube are given an orientation (shown by the red, green and blue circular arrows) by the rotation axes through each face. A simple renumbering of Chris's Diagonal family diagram gives solid and dashed corresponding to clockwise or anticlockwise oriented faces, resp.

A and B subfamilies vs. First and Second Corner families

Chris's distinguishes between **A** and **B** formations. For Improperish and Properish, **A** signifies partners across from each other, and **B** partners on the side (i.e. Becket). For the Diagonal formations, it has no obvious characterization. For the cube rotation model, the families divide nicely into two groups depending on the visible face of the cube have a diagonal of the tetrahedron on the first (\) or second (/) 'corners.'

The transformation diagram between the families in which an asymmetric pairs of dancers are swapped has a certain asymmetry with regards to **A** and **B** these are switched only between the Improperish and Diagonal families. However types (\) and (/) always switch when moving between any two different families.

First Member of Each Family

The choice of the beginning of each family is somewhat arbitrary, and Chris has chosen the most common Improperish and Properish formations as the first of each. Mathematically it would be nicer if the Properish family started with #3 as then all families would start on a clockwise 2nd corner diagonal face ☺.

Danceable transitions

Chris's paper discusses 3 types of transforms between families. The final two are the most simple in the cube model: the asymmetric moves ***Square Root of Trade Up/Down*** and ***Square Root of Trade Across***. They are just rotating the cube ¼ around the two axes parallel to the top face. These move all four dancers, but when done twice gives a trade (5 and 1, resp.). The first two involving one non-diagonal pair of dancers trading[2] are combinations of the cube rotations. For example "***Circle Left 1/2, Square Root of Trade Across"*** trades across the pair of dancers closest to the top of the set. Similarly the four involving three dancers circling can be given as combinations of cube rotations.

Examples:

Four in Line: Chris observed that having 1s step between the 2s can be viewed as Diagonal. In the usual improper hands four, having the facing couples pass thru, either the usual way or 1s stepping between the 2s, just before the dancers are all in a line of 4 if some dancers just get slightly ahead, one pair trades first, then shortly after the second trade is made. So in a sense this is the circle across move being done twice in a very short time frame.

4 face 4 is two parallel contra lines. Center or end directed moves are symmetric with respect to the overall formation, but asymmetric in each contra line half. *Center two pass thru and run around the ends* (MWSD terminology) on each line is an asymmetric move (of the type quarter double figure eight[3]) and puts each half into Diagonal formation. Doing this twice gives box the gnat across (= pass thru).
A template for a 4 face 4 dance with each half in Diagonal formation:
A1 (middle 4) LH star ½, (diag) Nbr Right Allemand ½, Center Left Allemand 1 ½
… play in diagonal formation keeping formation at end of A1
… centers pass thru, Partner Swing face your direction

Contra-like Dances

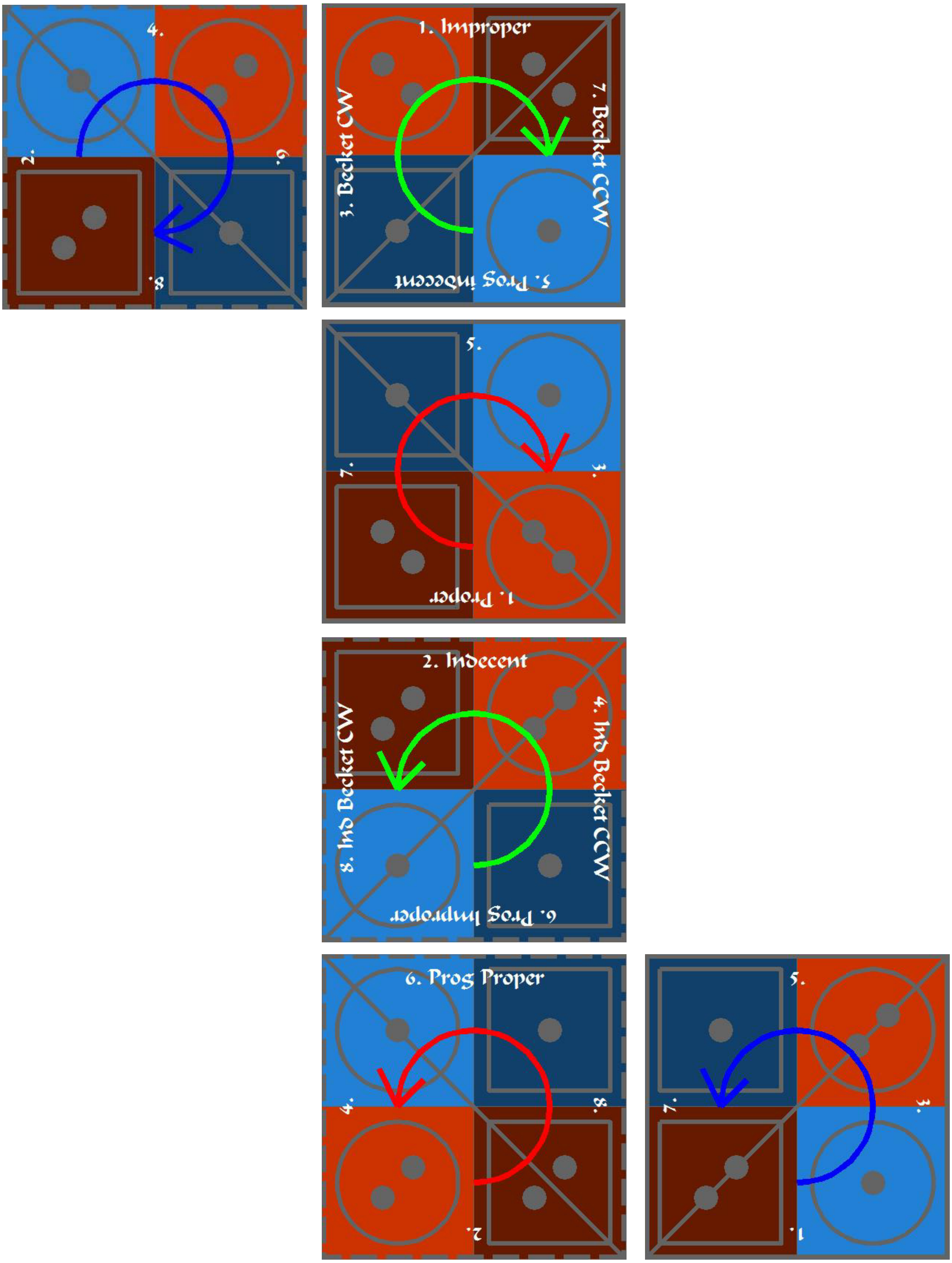
[2] Mathematician John Conway invented a two couple dance with rope using just circling together with one such move in which the ropes become entwined and then untangled.
[3] These moves were looked at in the context of MWSD. They are the four asymmetric two couple moves (among a total of ten moves) that can be used in a generalization of the Chicken Plucker routine.

Mixer “contra-like” dances. The formation structure applies to dances that look like a contra dance, but don’t keep partner or don’t progress. This implies that the progression aspect is not covered by this formation model.

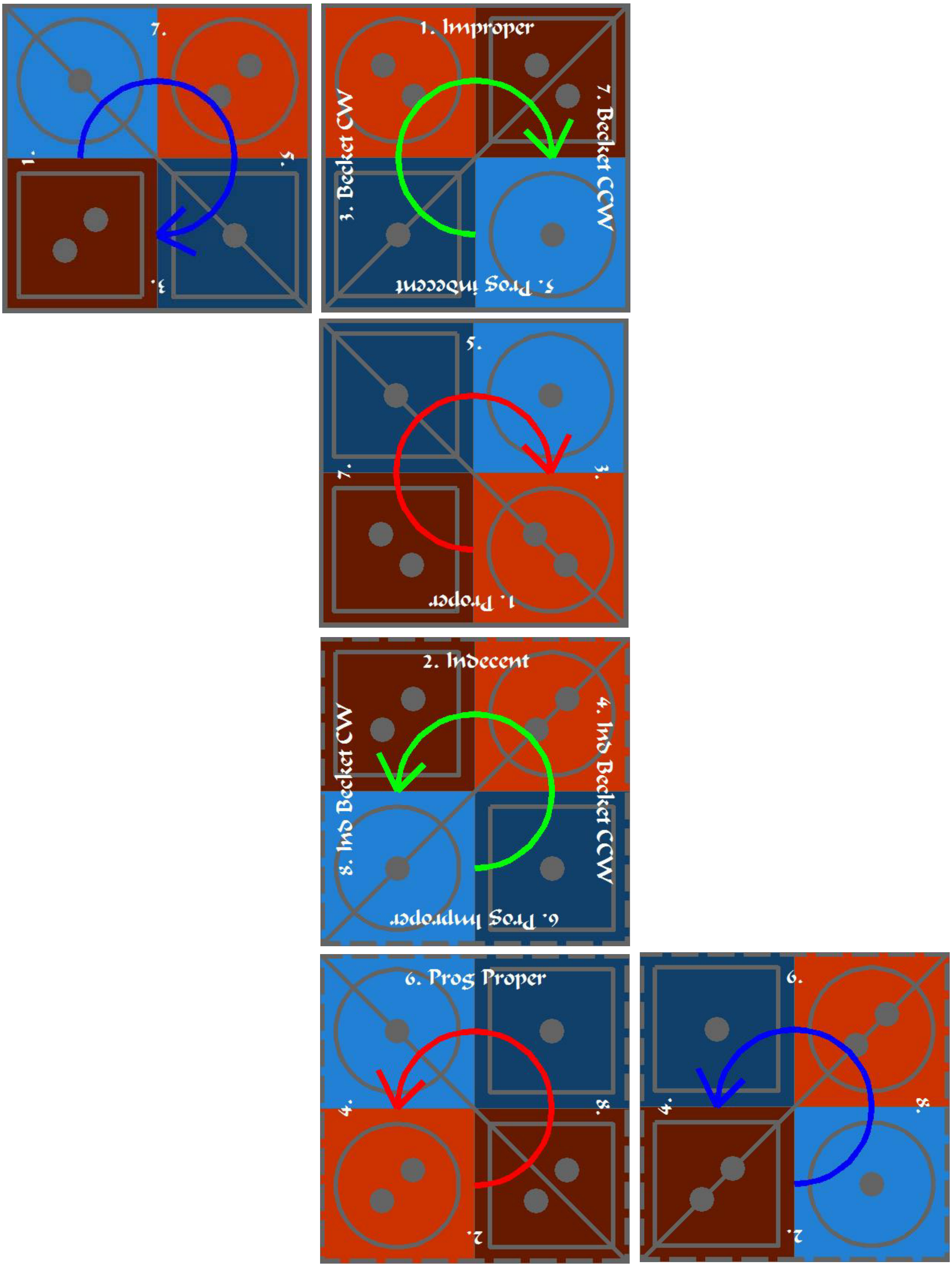
Here men and women progress at different rates, but the formation looks like a contra:
A1: N AR 1½ (next) N AL 1½ A2: LC, Bal sqth 2 B1: Sh B&SW B2: LC, Bal Sqth 2
5 5 5 3 4 5 5 3 4 5

555 34 55 345 in Chris’s notation, note Sqth2 is 4 or 15

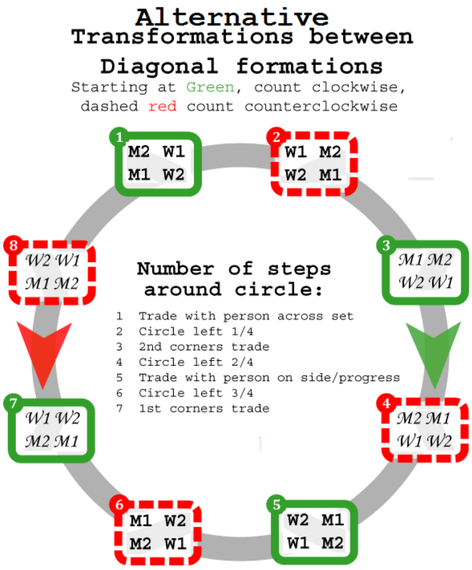
Make Your Own Formation Cube – Chris's Scheme



Make Your Own Formation Cube – Modified Scheme



Renumbering of Diagonal Diagram



Maths

Rotations of the Cube are isomorphic to S_4

Contra dance formations are permutations on four elements or elements of the group S_4. The three families correspond to three conjugate subgroups of order 8 each isomorphic to D_8, the Dihedral group of the symmetries of a square.

Geometric rotations of a cube (about three types of axis giving 4-fold, 3-fold and 2-fold rotation) are isomorphic to S_4, since the four main diagonals that connect opposite vertices are permuted under such rotations. The Contra Cube with faces colored by dancers has its final position the various formations. Since rotations are continuous, it is possible to move points representing dancers continuously through space. As the cube is rotated along its various axes, the points of intersection[4] of these lines with the plane[5] of a given face move thru the various permutations of the three families.

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[4] Another method of getting points on a square chose one end of each diagonal. Embedded in the cube are two tetrahedrons. Choosing the points of one of them give four points. Looking at the cube from the top, left side, or front, the shadows cast are the corners of a square. As the cube is rotated along its various axes, these shadows move thru the various permutations of the three families. However the correspondence is not so nice, mathematically it is not an isomorphism.

[5] In the matlab simulations, the rotations are shown dynamically, but the intersection of the moving diagonal line goes to infinity as the line becomes close to parallel with the face, and then returns from the opposite direction! Stereographically projecting these points in the plane onto the sphere tangent to the center of the cube and passing through the top four vertices give a compact dynamic view. This sphere has radius 3/2.

The family preserving and family changing moves

The period four rotations about the 3 axes perpendicular to the faces (colored red, green and blue) generate all the needed moves both to stay within a family[6] and all moves between them. Letting ***r***, ***g***, ***b*** correspond to the ¼ rotation in the directions indicated on the face of the cube. For example if the green clockwise marked face of the cube is on top, ***bb, g, grr, gg, rr, ggg, rrg*** give Chris's moves ***1, 2, …, 7*** resp. The non-diagonal transitions moves which trade two dancers are ***bgg, ggb, rgg*** and ***ggr.*** A transition moving just 3 dancers is ***gr***, while a final type is ***r*** and ***b***. Here the moves are describe with respect to the orientation and coloring of the cube, but an external (caller's) view point using moves like ***Circle Left, Square Root of Trade Across, Square Root of Trade Up/Down, Trade 1st corner, Trade 2nd corner*** as shown in the video.

Another viewpoint is that the commands of a contra dance cause dancers to move on the floor also induces commands for how a cube should rotate in space.

References

Bill Baritompa – Grid Squares
http://www.danceofmathematics.com/baritompa/contra/GridSquareJune2014.pdf

Bill Baritompa – General Chicken Plucker
http://www.danceofmathematics.com/baritompa/progsq/GeneralChickenPluckerOnly.pdf

John Conway "The Power of Mathematics" page 10
http://www.cs.toronto.edu/~mackay/abstracts/conway.html
http://www.danceofmathematics.com/movies/rangi2.avi

Chris Page "The 24 Contra Formations"
https://contrachoreography.wordpress.com/2013/11/06/the-twenty-four-duple-minor-formations/

[6] The three conjugate subgroups are $\langle r, g^2, b^2 \rangle$, $\langle g, b^2, r^2 \rangle$ and $\langle g, r^2, g^2 \rangle$. Let $m = rg$, $m^3 = 1$. Checking $mrm^{-1}=g$, $mgm^{-1}=b$ and $mbm^{-1}=r$ shows they are conjugate.